Tutorial 4

Q.1

Let $\mathcal{A} = \{ D \subseteq \mathbb{R}^3 : \partial D \text{ has measure zero and } D \text{ is bounded} \}$.

Find

$$\sup_{D \in \mathcal{A}} \int_D (1 - x^2 - y^2 - z^2)\, dV$$

Solution:

Note the integral is maximized on

$$D_0 = \{ (x, y, z) : 1 - x^2 - y^2 - z^2 \geq 0 \}$$

i.e. the unit ball.

We can then compute integral using spherical coordinates.

$$\sup_{D \in \mathcal{A}} \int_D (1 - x^2 - y^2 - z^2)\, dV$$

$$= \int_{D_0} (1 - x^2 - y^2 - z^2)\, dV$$

$$= \int_0^{\pi} \int_0^{2\pi} \int_0^1 (1 - \rho^2) \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= 2\pi \cdot 2 \cdot \frac{2}{15}$$

$$= \frac{8\pi}{15}$$

First moments and centre of mass

Let $\rho=\rho(x,y,z)$ be the (mass) density of an object represented by a region $D \subseteq \mathbb{R}^3$. (with ∂D measure zero, D bounded)

Mass: $M=\int_D \rho\, dV$ "$\rho dV = dm$: density × volume = mass"

First moment about the coefficient:

$$M_{yz}=\int_D x\rho\, dV$$

$$M_{xz}=\int_D y\rho\, dV$$

$$M_{xy}=\int_D z\rho\, dV$$

"Average mass of the object weighted by the distance by the coordinate plane"

The centre of mass of R is a point $(\bar{x},\bar{y},\bar{z})$, where

$$\bar{x}=\frac{M_{yz}}{M},\ \bar{y}=\frac{M_{xz}}{M},\ \bar{z}=\frac{M_{xy}}{M}$$

The 2D case is similar. The only difference is we have $M_x=\int_D y\rho\, dV$, $M_y=\int_D x\rho\, dV$, $\rho=\rho(x,y)$

$$\bar{x}=\frac{M_y}{M},\ \bar{y}=\frac{M_x}{M}.$$

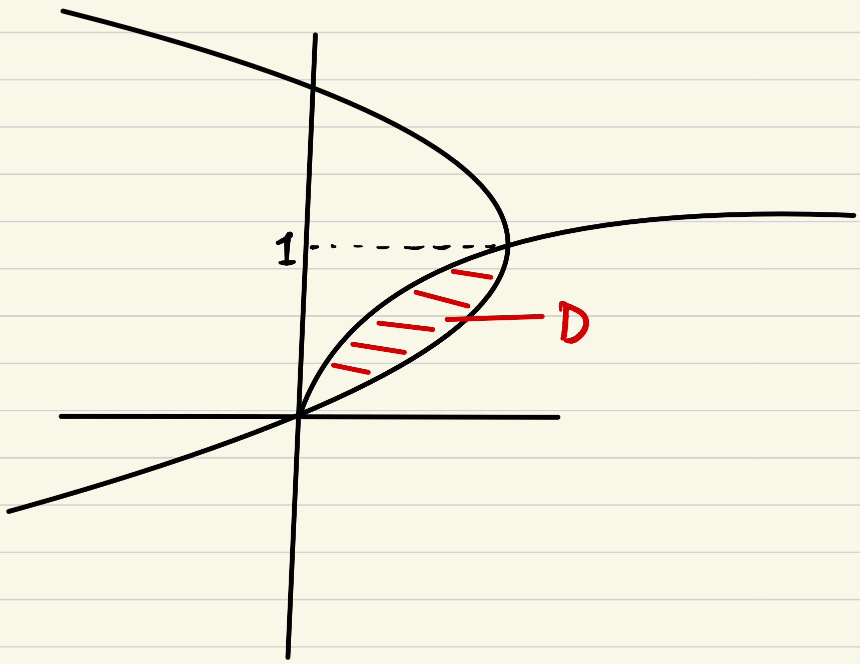
Remark: When $\rho\equiv 1$, the centre of mass is also called the centroid of R.

Q.2

Find the centre of mass of a thin plate bounded by the curves $x=y^2$ and $x=2y-y^2$ if the density at the point (x,y) is $\rho(x,y)=y+1$.

Solution:

The region:



$$y^2=2y-y^2 \Rightarrow y=0 \text{ or } 1$$

$$M=\int_D \rho \, dV$$

$$=\int_0^1\int_{y^2}^{2y-y^2}(1-y)\,dx\,dy$$

$$=\int_0^1 (1-y)(2y-2y^2)\,dy$$

$$=\frac{1}{2}$$

$$M_y = \int_D x\rho \, dV$$
$$= \int_0^1 \int_{y^2}^{2y-y^2} x(y+1)\, dx\, dy$$
$$= \frac{1}{2}\int_0^1 (y+1)\left((2y-y^2)^2 - y^4\right) dy$$
$$= \frac{8}{15}$$

$$M_x = \int_D y\rho \, dV$$
$$= \int_0^1 \int_{y^2}^{2y-y^2} y(y+1)\, dx\, dy$$
$$= \int_0^1 y(y+1)(2y-2y^2)\, dy$$
$$= \frac{4}{15}$$

$$\bar{x} = \frac{M_y}{M} = \frac{16}{15}$$

$$\bar{y} = \frac{M_x}{M} = \frac{8}{15}$$

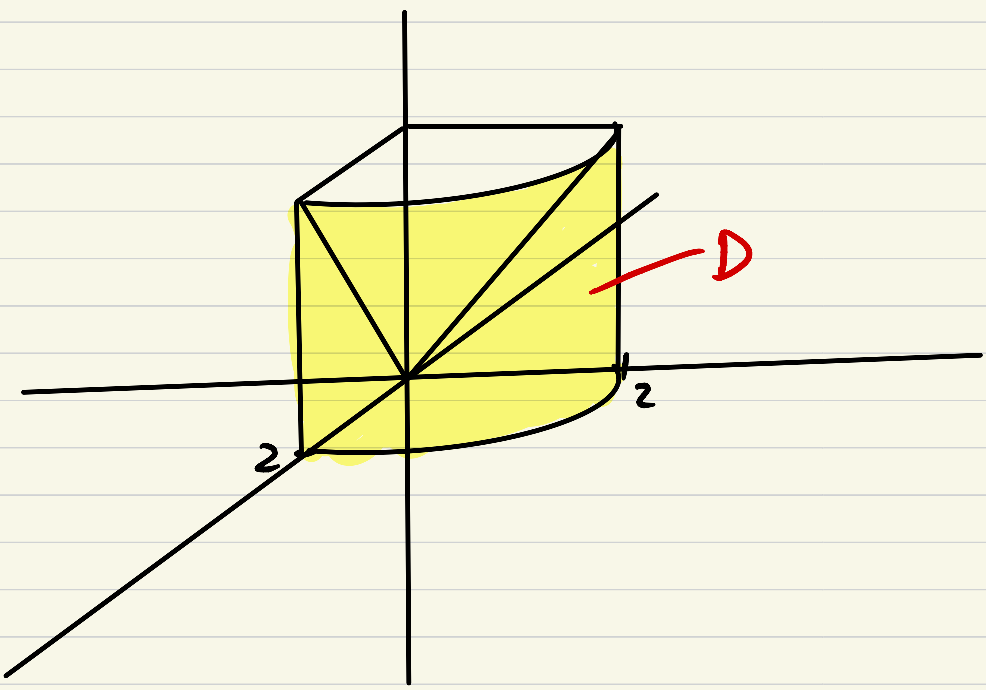
The centre of mass is at $\left(\frac{16}{15}, \frac{8}{15}\right)$.

Q.3

Find the centroid of the region in the 1st octant that is bounded above by the cone $z=\sqrt{x^2+y^2}$, below by the plane $z=0$, and on the sides by the cylinder $x^2+y^2=4$ and the planes $x=0$ and $y=0$.

Solution:

The region:



We use cylindrical coordinates here.

$$z=\sqrt{x^2+y^2}=\sqrt{r^2\cos^2\theta+r^2\sin^2\theta}=r$$

$0 \leq \theta \leq \frac{\pi}{2}$ $\qquad$ $dV = r\,dz\,dr\,d\theta$

$0 \leq r \leq 2$

$$M = \mathrm{vol}(D)$$

$$= \frac{1}{4} \times \left(\pi(2^2)(2) - \frac{1}{3}\pi(2^2)(2) \right)$$

$$= \frac{4\pi}{3}$$

$$M_{yz} = \int_D x \, dV$$

$$= \int_0^{\frac{\pi}{2}} \int_0^2 \int_0^r (r\cos\theta)(r) \, dz \, dr \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^2 r^3 \cos\theta \, dr \, d\theta$$

$$= 4$$

$$M_{xz} = \int_D y \, dV$$

$$= \int_0^{\frac{\pi}{2}} \int_0^2 \int_0^r (r\sin\theta)(r) \, dz \, dr \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^2 r^3 \sin\theta \, dr \, d\theta$$

$$= 4$$

$$M_{xy} = \int_D z \, dV$$

$$= \int_0^{\frac{\pi}{2}} \int_0^2 \int_0^r z r \, dz \, dr \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^2 \frac{r^3}{2} \, dr \, d\theta$$

$$= \pi$$

$$\bar{x} = \frac{M_{yz}}{M} = \frac{3}{\pi}$$

$$\bar{y} = \frac{M_{xz}}{M} = \frac{3}{\pi}$$

$$\bar{z} = \frac{M_{xy}}{M} = \frac{3}{4}$$

The centre of mass is at $\left(\frac{3}{\pi}, \frac{3}{\pi}, \frac{3}{4}\right)$.